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A multiplicative Schwarz iteration scheme for solving the linear complementarity problem with an H-matrix ☆

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ABSTRACT

In this paper, we consider an algebraic multiplicative Schwarz iteration scheme for solving the linear complementarity problem that involves an H_+-matrix. We show that the sequence generated by the multiplicative Schwarz iteration scheme converges to the unique solution of the problem without any restriction on the initial point. For different overlapping sizes, the convergence rate of the proposed method is analyzed in an algebraic setting. Moreover, we establish monotone convergence of the proposed method under appropriate conditions. Numerical results show that efficiency can be achieved by the multiplicative Schwarz iteration scheme.

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1. Introduction

We consider the following finite-dimensional linear complementarity problem (LCP):

$$\begin{array}{ll} \text{find} & x \in \mathbb{R}^n, \\ \text{such that} & x \geqslant \phi, \quad Ax - F \geqslant 0, \quad (x-\phi)^T(Ax-F) = 0, \end{array} \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a given matrix, and $\phi, F \in \mathbb{R}^n$ are given vectors. If all components of vector ϕ are 0, then problem (1.1) reduces to the LCP shown in [1–3]. If all components of vector ϕ become $-\infty$, then problem (1.1) reduces to the system of linear equations

$$Ax = F. \tag{1.2}$$

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In this paper, we assume that A is an H-matrix with positive diagonals, i.e., A is an H_+-matrix; see [3]. A nonsingular matrix A having all nonpositive off-diagonal entries is called an M-matrix if the inverse is (entry-wise) nonnegative, i.e., $A^{-1} \geqslant 0$; see, e.g., [8]. For any matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, its comparison matrix $\langle A\rangle = (\alpha_{ij})$ is defined by

$$\alpha_{ii} = |a_{ii}|, \quad \alpha_{ij} = -|a_{ij}|, \ i \neq j.$$

A matrix A is said to be an H-matrix if $\langle A\rangle$ is an M-matrix. H-matrix was introduced as a generalization of M-matrix. It appears in many applications, e.g., when discretizing certain nonlinear parabolic operators using high order finite elements and sufficiently small time steps [12].

Recently, various Schwarz iterative methods for solving finite dimensional variational inequalities as well as complementarity problems have been presented [10,18,23,24,26,27]. This kind of methods is amenable to implement. Moreover, the convergence rate will not be deteriorated with the refinement of the mesh when applied to discretized differential equations. Theory and numerical experiments have shown that the latter advantage is still maintained when the methods are used to solve discretized variational inequalities with an elliptic differential operator [24,26]. Generally, there are two ways to study convergence of Schwarz methods for solving LCPs. One is to prove that the method generates a minimizing sequence for some objective function. In this case, the matrix A is often supposed to be symmetric and positive definite. The other way is to prove that the method produces a monotone sequence starting from a super-solution or a lower-solution of the problem. Convergence theorems established in the latter way are often based on the assumption that matrix A is an M-matrix.

Up to now, there is no general convergence theory for the LCP (1.1) with an H_+-matrix by using the multiplicative Schwarz method. The purpose of this paper is to apply a multiplicative Schwarz iteration scheme to solve the LCP (1.1) when A is an H_+-matrix. The scheme is an extension of the multiplicative Schwarz iteration scheme for solving the linear equation (1.2), which was proposed by Bru et al. [9]. We show that the sequence generated by the multiplicative Schwarz iteration scheme converges to the unique solution of the problem without any restriction on the initial point. For different overlapping sizes, the convergence rate of the proposed method is analyzed in an algebraic setting. We show that the proposed method generates a monotone sequence of iterates if the coefficient matrix A is an M-matrix and the initial point is a super-solution of the problem.

The paper is organized as follows: in Section 2, we propose a multiplicative Schwarz iteration scheme for solving the LCP (1.1) when A is an H_+-matrix. In Section 3, we give some basic properties of the proposed method. In Section 4, we estimate the weighted max-norm bounds for iteration errors and establish global convergence of the multiplicative Schwarz iteration scheme. In Section 5, we analyze the convergence rate of the proposed method for different overlapping sizes. In Section 6, we show monotone convergence of the proposed method under appropriate conditions. Finally, in Section 7, we give some numerical results to verify the efficiency of the multiplicative Schwarz iteration scheme.

2. Multiplicative Schwarz iteration scheme

In this section, we propose an multiplicative Schwarz iteration scheme for solving the LCP (1.1) when A is an H_+-matrix. First, we give some notations that will be used throughout the paper. Let $V_i, i = 1, \ldots, m$, be subspaces of $\mathbb{R}^n$ such that

$$\sum_{i=1}^{m} V_i \equiv \{v \in V : v = v_1 + v_2 + \cdots + v_m,\ v_i \in V_i\ (i = 1, \ldots, m)\} = \mathbb{R}^n. \tag{2.1}$$

That is to say, the bases of the subspaces $V_i, i = 1, \ldots, m$, span the space $\mathbb{R}^n$. One step of the multiplicative Schwarz iteration scheme consists of the following process: restrict the current residual and solve the local problem on the subspace V_i, prolongate the approximation of the error and add the error to the correction. Let $n_i = \dim(V_i)$ be the dimensions of subspaces $V_i, i = 1, \ldots, m$. We consider both overlapping subdomains and nonoverlapping subdomains, which correspond to the cases $\sum_{i=1}^{m} n_i > n$ and $\sum_{i=1}^{m} n_i = n$, respectively. For simplicity, we identify V_i with $\mathbb{R}^{n_i}$. Let $R_i : \mathbb{R}^n \longrightarrow \mathbb{R}^{n_i}$ be the restriction operator. In our context, R_i is an $n_i \times n$ matrix with rank n_i. Its transpose $R_i^T : \mathbb{R}^{n_i} \longrightarrow \mathbb{R}^n$, is a prolongation operator. Let $A_i = R_i A R_i^T$ denote the restriction of A to V_i. Moreover, we choose the bases of V_i appropriately such that the image of the bases of V_i under the prolongation operator R_i^T are linearly independent unit elements in $\mathbb{R}^n$. In other words, the columns of R_i^T consist of columns of the $n \times n$ identity matrix. Formally, such a matrix R_i can be expressed as

$$R_i = [I_i \ 0]\pi_i \geqslant 0, \tag{2.2}$$

where I_i is the $n_i \times n_i$ identity matrix and π_i is some $n \times n$ permutation matrix. In this case, matrix A_i is an $n_i \times n_i$ principal submatrix of A. Let $\pi \in \mathbb{R}^{n\times n}$ be a permutation matrix, we denote by $A_\pi = \pi A \pi^T$. Noting that

$$A_i = R_i A R_i^T = [I_i \ 0] A_{\pi_i} \begin{bmatrix} I_i \\ 0 \end{bmatrix} \in \mathbb{R}^{n_i \times n_i} \tag{2.3}$$

is the $n_i \times n_i$ leading principal submatrix of A_{π_i}, we can represent the matrix A_{π_i} in the form

$$A_{\pi_i} = \begin{bmatrix} A_i & G_i \\ H_i & A_{i_c} \end{bmatrix}. \tag{2.4}$$

Similarly, we denote by $x_\pi = \pi x \in \mathbb{R}^n$, $F_\pi = \pi F \in \mathbb{R}^n$ and $\phi_\pi = \pi \phi \in \mathbb{R}^n$. Then we can represent the vectors x_{π_i}, F_{π_i} and ϕ_{π_i} as

$$x_{\pi_i} = \begin{bmatrix} u_i \\ u_{i_c} \end{bmatrix}, \quad F_{\pi_i} = \begin{bmatrix} f_i \\ f_{i_c} \end{bmatrix}, \quad \phi_{\pi_i} = \begin{bmatrix} \varphi_i \\ \varphi_{i_c} \end{bmatrix},$$

where $u_i = R_i x \in \mathbb{R}^{n_i}$, $f_i = R_i F \in \mathbb{R}^{n_i}$ and $\varphi_i = R_i \phi \in \mathbb{R}^{n_i}$.

Moreover, if A is an H-matrix, in a way similar to (2.3), we have

$$\langle A \rangle_i = R_i \langle A \rangle R_i^T \in \mathbb{R}^{n_i \times n_i}.$$

It is easy to see that $\langle A \rangle_i = \langle A_i \rangle$, and any principal submatrix of an M-matrix is also an M-matrix [8]. We have the following useful result.

Lemma 2.1 [9]. *If $A \in \mathbb{R}^{n\times n}$ is an H-matrix, then any principal submatrix of A, and any of symmetric permutation is an H-matrix. In particular, the matrix A_i given by (2.3) and A_{π_i} given by (2.4) are H-matrices.*

Let x^0 be an initial approximation to the solution of problem (1.1). Then the multiplicative Schwarz iteration scheme for problem (1.1) is as follows:

Algorithm 1 (**Multiplicative Schwarz iteration scheme**)

Step 1: Let $z^{0,0} = x^0$ be an arbitrary vector, and set $k := 0$.

Step 2: Given $z^{k,0} = x^k$, for $i = 1, 2, \ldots, m$, do the following sub-steps:

Step 2.1 (restriction): Restrict the current residual $F - Az^{k,i-1}$ and the vector $\phi - z^{k,i-1}$ as

$$R_e^{k,i} = R_i(F - Az^{k,i-1}), \tag{2.5}$$
$$\phi^{k,i} = R_i(\phi - z^{k,i-1}). \tag{2.6}$$

Let $x^{k,i}$ be the solution of the following problem:

$$\begin{cases} x^{k,i} \geqslant \phi^{k,i}, \\ A_i x^{k,i} \geqslant R_e^{k,i}, \\ (x^{k,i} - \phi^{k,i})^T (A_i x^{k,i} - R_e^{k,i}) = 0, \end{cases} \tag{2.7}$$

where $A_i = R_i A R_i^T$.

Step 2.2 (prolongation): Prolongate the approximation by

$$x_e^{k,i} = R_i^T x^{k,i}. \tag{2.8}$$

Step 2.3 (correction): Correct $z^{k,i-1}$ to get

$$z^{k,i} = z^{k,i-1} + \theta_i x_e^{k,i}, \tag{2.9}$$

where θ_i is a given positive weight.

Step 3: Let $x^{k+1} = z^{k,m}$. If $x^{k+1} = x^k$, then stop. Otherwise, set $k := k + 1$ and return to Step 2.

Remark 2.1. For the system of linear equations (1.2), if $\theta_1 = \theta_2 = \cdots = \theta_m = 1$, then the above multiplicative Schwarz iteration reduces to the multiplicative Schwarz iteration scheme proposed in [9].

The following concept will play an important role in the subsequent analysis.

Definition 2.2 [20]. Let $\omega \in \mathbb{R}^n$ be a positive vector. For a vector $y \in \mathbb{R}^n$, the weighted max-norm is defined by

$$\|y\|_\omega = \max_{1\leqslant j\leqslant n} \left|\frac{y_j}{\omega_j}\right|.$$

For a matrix $A \in \mathbb{R}^{n\times n}$, the weighted max-norm is defined by

$$\|A\|_\omega = \sup_{\|y\|_\omega=1} \{\|Ay\|_\omega : y \in \mathbb{R}^n\}.$$

Obviously, if $\omega = (1, \ldots, 1)^T$, then the weighted max-norm reduces to the usual maximum norm.

We say that $A = M - N$ is a splitting if M is nonsingular. The splitting is regular if $M^{-1} \geqslant 0$ and $N \geqslant 0$; it is weak regular if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$; and nonnegative if $M^{-1} \geqslant 0$, $M^{-1}N \geqslant 0$ and $NM^{-1} \geqslant 0$; see [8,25]; this splitting is called an M-splitting if M is an M-matrix and $N \geqslant 0$; it is called H-compatible if $\langle A\rangle = \langle M\rangle - |N|$ [14]. It is obvious that an M-splitting is a regular splitting. Hence, an M-splitting is also a weak regular splitting and a nonnegative splitting. From [14], if the splitting is an H-compatible splitting and A is an H-matrix, then $\langle A\rangle = \langle M\rangle - |N|$ is an M-splitting. So we have the following result.

Lemma 2.3. *If $A \in \mathbb{R}^{n\times n}$ is an H-matrix and $A = M - N$ is an H-compatible splitting, then $\langle A\rangle = \langle M\rangle - |N|$ is a regular splitting.*

The following result, which can be found, e.g., in [7], shows that given an iteration matrix, there exists a unique splitting.

Lemma 2.4. *Let A and T be square matrices such that A and $I - T$ are nonsingular. Then, there exists a unique pair of matrices (B, C) such that B is nonsingular, $T = B^{-1}C$ and $A = B - C$. The matrices are $B = A(I - T)^{-1}$ and $C = B - A = A((I - T)^{-1} - I)$.*

3. Preliminaries

In this section, we prove some useful properties about Algorithm 1. For each $i = 1, \ldots, m$, we construct diagonal matrices $E_i \in \mathbb{R}^{n\times n}$ associated with R_i in (2.2) as follows:

$$E_i = R_i^T R_i. \tag{3.1}$$

These diagonal matrices have ones on the diagonal in every row where R_i^T has nonzeros. We further assume that if S_i is the set of indices of the rows of the identity that are rows of R_i, then

$$\bigcup_{i=1}^{m} S_i = S = \{1, 2, \ldots, n\}. \tag{3.2}$$

In other words, each variable is in at least one set S_i. This is equivalent to saying that $\sum_{i=1}^m E_i \geqslant I$, with equality if and only if there is no overlap. Note that in the case of overlapping blocks, we have here that each diagonal entry of $\sum_{i=1}^m E_i$ is greater than or equal to one, which implies nonsingularity; see [4].

In the following, we give the existence and uniqueness of the solution of the LCP (1.1) shown in [3,11].

Lemma 3.1 [3,11]. *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix. Then the LCP (1.1) has a unique solution for any $F \in \mathbb{R}^n$.*

Since A_i is also an H_+-matrix from Lemmas 2.1 and 3.1, we know that problem (2.7) also has a unique solution.

The following lemma shows that if, at some step k, $z^{k,i-1}$ coincides with the unique solution of problem (1.1), then $0 \in \mathbb{R}^{n_i}$ is the unique solution of the problem (2.7).

Lemma 3.2. *Let $\bar{x}$ be the unique solution of (1.1). If $z^{k,i-1} = \bar{x}$, then we have $x^{k,i} = 0$ for some $i \in \{1, 2 \ldots, m\}$.*

Proof. Since $\bar{x} - \phi \geqslant 0$, $A\bar{x} - F \geqslant 0$ and $(\bar{x} - \phi)^T(A\bar{x} - F) = 0$, it follows from the nonnegativity of R_i that

$$\begin{cases} 0 - \phi^{k,i} = R_i(\bar{x} - \phi) \geqslant 0, \\ A_i 0 - R_e^{k,i} = R_i(A\bar{x} - F) \geqslant 0. \end{cases} \tag{3.3}$$

Multiplying these two inequalities and noting $E_i = R_i^T R_i \leqslant I$, we have

$$0 \leqslant (0 - \phi^{k,i})^T (A_i 0 - R_e^{k,i}) = (\bar{x} - \phi)^T R_i^T R_i (A\bar{x} - F) \leqslant (\bar{x} - \phi)^T (A\bar{x} - F) = 0$$

and, hence,

$$(0 - \phi^{k,i})^T (A_i 0 - R_e^{k,i}) = 0. \tag{3.4}$$

It follows from (3.3) and (3.4) that $x^{k,i} = 0$ is a solution of (2.7), which is unique since A_i is an H_+-matrix. □

Lemma 3.3. *Let $z_{\pi_i}^{k,i-1} = \begin{bmatrix} u_i^{k,i-1} \\ u_{i_c}^{k,i-1} \end{bmatrix}$ with $u_i^{k,i-1} = R_i z^{k,i-1} \in \mathbb{R}^{n_i}$, and $y^{k,i} \in \mathbb{R}^{n_i}$ be given by $y^{k,i} = u_i^{k,i-1} + x^{k,i}$. Then $y^{k,i}$ is the solution of the following LCP on $\mathbb{R}^{n_i}$:*

$$\begin{cases} y \geqslant \varphi_i, \\ A_i y - F^{k,i-1} \geqslant 0, \\ (y - \varphi_i)^T (A_i y - F^{k,i-1}) = 0, \end{cases} \tag{3.5}$$

where $\varphi_i = R_i\phi$, $F^{k,i-1} = f_i - G_i u_{i_c}^{k,i-1}$ and $f_i = R_i F$.

Proof. By the definition of $y^{k,i}$, we have

$$y^{k,i} - \varphi_i = u_i^{k,i-1} + x^{k,i} - \varphi_i = x^{k,i} - R_i(\phi - z^{k,i-1}) = x^{k,i} - \phi^{k,i}.$$

Since $\pi_i^T \pi_i = I$, it follows from (2.4) that

$$\begin{aligned} A_i y^{k,i} - F^{k,i-1} &= A_i x^{k,i} + A_i u_i^{k,i-1} + G_i u_{i_c}^{k,i-1} - f_i \\ &= A_i x^{k,i} + [A_i G_i] z_{\pi_i}^{k,i-1} - R_i F \\ &= A_i x^{k,i} + [I_i 0] A_{\pi_i} z_{\pi_i}^{k,i-1} - R_i F \\ &= A_i x^{k,i} + [I_i 0] \pi_i A \pi_i^T \pi_i z^{k,i-1} - R_i F \\ &= A_i x^{k,i} + R_i (A z^{k,i-1} - F) \\ &= A_i x^{k,i} - R_e^{k,i}. \end{aligned}$$

Consequently, (3.5) follows from (2.7). □

Lemmas 3.2 and 3.3 imply the following corollary.

Corollary 3.4. *Let $\bar{x}$ be the solution of* (1.1). *Then $\bar{u}_i = R_i\bar{x}$ is the unique solution of the following LCP on $\mathbb{R}^{n_i}$:*

$$\begin{cases} y \geqslant \varphi_i, \\ A_i y - F^{*,i} \geqslant 0, \\ (y-\varphi_i)^T(A_i y - F^{*,i}) = 0, \end{cases} \tag{3.6}$$

where $\varphi_i = R_i\phi$, $F^{*,i} = f_i - G_i\bar{u}_{i_c}$ and $f_i = R_iF$.

4. Weighted max-norm bounds

In this section, we estimate the weighted max-norm bounds for the iteration errors of Algorithm 1. First, we give some notations. Given a matrix $A = (a_{ij})$, we define the matrix $|A| = (|a_{ij}|)$. It follows that $|A| \geqslant 0$ and $|AB| \leqslant |A||B|$ for any matrices A and B of compatible size. Note that

$$A_{i_c} = [0\ I_{i_c}]\pi_i A \pi_i^T [0\ I_{i_c}]^T, \tag{4.1}$$

where I_{i_c} is the $(n-n_i)\times(n-n_i)$ identity matrix. Let

$$M_i = \pi_i^T \begin{bmatrix} A_i & 0 \\ 0 & B_{i_c} \end{bmatrix} \pi_i, \tag{4.2}$$

where B_{i_c} is some $(n-n_i)\times(n-n_i)$ nonsingular matrix such that

$$|B_{i_c} - A_{i_c}| = \langle B_{i_c}\rangle - \langle A_{i_c}\rangle. \tag{4.3}$$

In fact, this condition gives a lot of freedom in choosing B_{i_c}. In [7,15], different choices were $B_{i_c} = A_{i_c}$ or $B_{i_c} = D_{i_c} = \mathrm{diag}(A_{i_c})$. These choices clearly satisfy our condition (4.3).

It follows from the matrices in (3.1) and (4.2) that

$$E_i M_i^{-1} = R_i^T A_i^{-1} R_i \tag{4.4}$$

and, hence,

$$E_i\langle M_i\rangle^{-1} = R_i^T\langle A_i\rangle^{-1}R_i. \tag{4.5}$$

The following lemmas are useful.

Lemma 4.1 [9]. *Let A be an H-matrix and the matrices M_i be of the form* (4.2), *satisfying* (4.3). *Then, $A = M_i - N_i, i = 1,\ldots,m$, are H-compatible splittings.*

Obviously, the above splittings $\langle A\rangle = \langle M_i\rangle - |N_i|$, $i = 1,\ldots,m$, are regular (and thus weak regular and nonnegative) from Lemmas 4.1 and 2.3.

Remark 4.1. The Schwarz iteration method is related to the multisplitting iteration method [3], as special choices of the weighting and the splitting matrices in a multisplitting can naturally lead to a Schwarz iteration method. Hence, the collection of triples $\{(\theta_iE_i, M_i, N_i)\}_{i=1}^m$ can be thought of as a multiplicative multisplitting of A, in analogy with the standard (additive) multisplitting of a matrix in the sense of [3]. And the multiplicative Schwarz iteration method can be thought of as a chaotic multisplitting iteration method [2]. Hence, Algorithm 1 can be also thought of as a chaotic multisplitting iteration method.

Lemma 4.2 [15]. *Let P be a matrix, ω be a positive vector, and γ be a positive scalar such that*

$$|P|\omega \leqslant \gamma\omega. \tag{4.6}$$

Then $\|P\|_\omega \leqslant \gamma$. In particular, $\|Px\|_\omega \leqslant \gamma\|x\|_\omega$ holds for all x. Moreover, if strict inequality holds in (4.6), *then we have $\|P\|_\omega < \gamma$.*

Lemma 4.3. *Let $\bar{x}$ be the unique solution of* (1.1) *and $\bar{x}_{\pi_i} = [\bar{u}_i \bar{u}_{i_c}]^T$, $i = 1, \ldots, m$. Denote $y^{k,i} = u_i^{k,i-1} + x^{k,i}$ and $\bar{y}^{*,i} = \bar{u}_i$. Then*

$$\langle A_i \rangle |y^{k,i} - \bar{y}^{*,i}| \leqslant |G_i||u_{i_c}^{k,i-1} - \bar{u}_{i_c}|. \tag{4.7}$$

Proof. Following the proof of Theorem 3.1 in [3], we can verify (4.7) by componentwise. Consider an arbitrary index *j*. We first assume that

$$|y^{k,i} - \bar{y}^{*,i}|_j = (y^{k,i} - \bar{y}^{*,i})_j,$$

which means that

$$(y^{k,i} - \bar{y}^{*,i})_j \geqslant 0.$$

Thus, if $y_j^{k,i} = \phi_j$, then $\bar{y}_j^{*,i} = \phi_j$. Hence, (4.7) holds for the *j*th component, since the left-hand side is nonpositive while the right-hand side is nonnegative.

If $y_j^{k,i} > \phi_j$, then by Lemma 3.3, we have

$$(A_i y^{k,i} - f_i + G_i u_{i_c}^{k,i-1})_j = 0. \tag{4.8}$$

Furthermore, by Corollary 3.4 we have

$$(A_i \bar{y}^{*,i} - f_i + G_i \bar{u}_{i_c})_j \geqslant 0. \tag{4.9}$$

Thus, by subtracting (4.9) from (4.8), we get

$$(A_i(y^{k,i} - \bar{y}^{*,i}))_j \leqslant -(G_i(u_{i_c}^{k,i-1} - \bar{u}_{i_c}))_j \leqslant (|G_i||u_{i_c}^{k,i-1} - \bar{u}_{i_c}|)_j.$$

Note that

$$(A_i(y^{k,i} - \bar{y}^{*,i}))_j \geqslant (\langle A_i \rangle |y^{k,i} - \bar{y}^{*,i}|)_j,$$

as A_i is an H_+-matrix. So we have

$$(\langle A_i \rangle |y^{k,i} - \bar{y}^{*,i}|)_j \leqslant (|G_i||u_{i_c}^{k,i-1} - \bar{u}_{i_c}|)_j. \tag{4.10}$$

We next assume that

$$|y^{k,i} - \bar{y}^{*,i}|_j = (\bar{y}^{*,i} - y^{k,i})_j.$$

In this case, we have

$$(y^{k,i} - \bar{y}^{*,i})_j \leqslant 0.$$

In a similar fashion, we can establish the same inequality (4.10). Thus inequality (4.7) holds. □

The following lemma gives an estimate of the iteration errors.

Lemma 4.4. *Let θ_i be positive constants satisfying $0 < \theta_i \leqslant 1$ and $\varepsilon^{k,i} \triangleq z^{k,i} - \bar{x}$ for $i = 0, 1, \ldots, m$, where $\bar{x}$ is the unique solution of* (1.1). *Then $|\varepsilon^{k,i}| \leqslant (I - \theta_i E_i \langle M_i \rangle^{-1} \langle A \rangle)|\varepsilon^{k,i-1}|$ for $i = 1, \ldots, m$.*

Proof. We deduce from (2.9) that

$$\begin{aligned} 0 &\leqslant |\varepsilon^{k,i}| = |z^{k,i} - \bar{x}| \\ &= |z^{k,i-1} + \theta_i x_e^{k,i} - \bar{x}| \\ &= |\varepsilon^{k,i-1} + \theta_i R_i^T x^{k,i}| \\ &= |(1-\theta_i)\varepsilon^{k,i-1} + \theta_i (R_i^T x^{k,i} + \varepsilon^{k,i-1})| \\ &= \left| (1-\theta_i)\varepsilon^{k,i-1} + \theta_i \pi_i^T \left(\begin{bmatrix} x^{k,i} \\ 0 \end{bmatrix} + \begin{bmatrix} u_i^{k,i-1} - \bar{u}_i \\ u_{i_c}^{k-1} - \bar{u}_{i_c} \end{bmatrix} \right) \right| \end{aligned}$$

$$
\begin{aligned}
&\leqslant |(1-\theta_i)\varepsilon^{k,i-1}| + \left|\theta_i\pi_i^T \begin{bmatrix} |y^{k,i} - \bar{y}^{*,i}| \\ |u_{i_c}^{k,i-1} - \bar{u}_{i_c}| \end{bmatrix}\right| \\
&\leqslant |1-\theta_i||\varepsilon^{k,i-1}| + \left|\theta_i\pi_i^T \begin{bmatrix} \langle A_i\rangle^{-1}|G_i||u_{i_c}^{k,i-1} - \bar{u}_{i_c}| \\ |u_{i_c}^{k,i-1} - \bar{u}_{i_c}| \end{bmatrix}\right|,
\end{aligned}
$$

where $y^{k,i} = u_i^{k,i-1} + x^{k,i}$ and $\bar{y}^{*,i} = \bar{u}_i$, the third equality follows from (2.2), and the last inequality follows from (4.7). Since $0 < \theta_i \leqslant 1$, we deduce from the above formula that

$$
\begin{aligned}
0 &\leqslant |\varepsilon^{k,i}| \\
&\leqslant (1-\theta_i)|\varepsilon^{k,i-1}| + \theta_i\pi_i^T \begin{bmatrix} 0 & \langle A_i\rangle^{-1}|G_i| \\ 0 & I_{i_c} \end{bmatrix} |\varepsilon_{\pi_i}^{k,i-1}| \\
&= |\varepsilon^{k,i-1}| - \theta_i\pi_i^T|\varepsilon_{\pi_i}^{k,i-1}| + \theta_i\pi_i^T \begin{bmatrix} 0 & \langle A_i\rangle^{-1}|G_i| \\ 0 & I_{i_c} \end{bmatrix} |\varepsilon_{\pi_i}^{k,i-1}| \\
&= |\varepsilon^{k,i-1}| + \theta_i\pi_i^T \begin{bmatrix} -I_i & \langle A_i\rangle^{-1}|G_i| \\ 0 & 0 \end{bmatrix} |\varepsilon_{\pi_i}^{k,i-1}| \\
&= |\varepsilon^{k,i-1}| - \theta_i R_i^T\langle A_i\rangle^{-1}[\langle A_i\rangle - |G_i|]|\varepsilon_{\pi_i}^{k,i-1}| \\
&= |\varepsilon^{k,i-1}| - \theta_i R_i^T\langle A_i\rangle^{-1}[I_i 0]\langle A\rangle_{\pi_i}|\varepsilon_{\pi_i}^{k,i-1}| \\
&= |\varepsilon^{k,i-1}| - \theta_i R_i^T\langle A_i\rangle^{-1}[I_i 0]\pi_i\langle A\rangle\pi_i^T\pi_i|\varepsilon^{k,i-1}| \\
&= |\varepsilon^{k,i-1}| - \theta_i R_i^T\langle A_i\rangle^{-1}R_i\langle A\rangle|\varepsilon^{k,i-1}| \\
&= (I - \theta_i E_i\langle M_i\rangle^{-1}\langle A\rangle)|\varepsilon^{k,i-1}|,
\end{aligned}
$$

where the third and sixth equalities follow from (2.2), the fourth equality follows from (2.4) and the last equality follows from (4.5). $\square$

Lemma 4.5. *Let θ_i be positive constants satisfying $0 < \theta_i \leqslant 1, i = 1, \ldots, m$, and $\varepsilon^k \triangleq x^k - \bar{x}$, where $\bar{x}$ is the unique solution of* (1.1). *Then we have*

$$0 \leqslant |\varepsilon^{k+1}| \leqslant T_\theta|\varepsilon^k|,$$

where $T_\theta = \prod_{i=m}^{1}(I - \theta_i E_i\langle M_i\rangle^{-1}\langle A\rangle)$ is a nonnegative matrix. Moreover, for any vector $\omega = \langle A\rangle^{-1}e$ with $e > 0$, there exists a scalar $\gamma \in (0, 1)$ such that

$$\|T_\theta\|_\omega \leqslant \gamma, \tag{4.11}$$

and $\rho(T_\theta) \leqslant \|T_\theta\|_\omega \leqslant \gamma$, where $\rho(B)$ denotes the spectral radius of a matrix B. Hence,

$$\|\varepsilon^{k+1}\|_\omega \leqslant \|T_\theta\varepsilon^k\|_\omega \leqslant \gamma\|\varepsilon^k\|_\omega.$$

Proof. By Lemma 4.4, we have

$$
\begin{aligned}
|\varepsilon^{k+1}| &= |x^{k+1} - \bar{x}| = |z^{k,m} - \bar{x}| = |\varepsilon^{k,m}| \\
&\leqslant (I - \theta_m E_m\langle M_m\rangle^{-1}\langle A\rangle)|\varepsilon^{k,m-1}| \\
&\leqslant \prod_{i=m}^{1}(I - \theta_i E_i\langle M_i\rangle^{-1}\langle A\rangle)|\varepsilon^{k,0}| \\
&= T_\theta|\varepsilon^k|.
\end{aligned}
$$

In order to show $\|T_\theta\|_\omega \leqslant \gamma$, we only need to show that $T_\theta \geqslant 0$ and $T_\theta\omega < \omega$. Clearly, $T_\theta \geqslant 0$, as for $i = 1, \ldots, m$,

$$\begin{aligned}
&I - \theta_i E_i \langle M_i \rangle^{-1} \langle A \rangle \\
&= I - \theta_i E_i + \theta_i E_i (I - \langle M_i \rangle^{-1} \langle A \rangle) \\
&= I - \theta_i E_i + \theta_i E_i \langle M_i \rangle^{-1} |N_i| \\
&\geqslant 0,
\end{aligned}$$

where the second equality follows from Lemma 4.1, and the inequality follows from $0 \leqslant E_i \leqslant I$, $0 < \theta_i \leqslant 1$ and $\langle M_i \rangle^{-1} |N_i| \geqslant 0$.

Next, we show that $T_\theta \omega < \omega$ with $\omega = \langle A \rangle^{-1} e$, where $e > 0$. To begin with, note that

$$\omega_1 := (I - \theta_1 E_1 \langle M_1 \rangle^{-1} \langle A \rangle)\omega = \omega - \theta_1 E_1 \langle M_1 \rangle^{-1} e \geqslant 0.$$

Hence, $0 \leqslant \omega_1 \leqslant \omega$, with strict inequality in the components corresponding to S_1. In other words, denoting with $(\omega_1)_i$ the ith component of ω_1, we have

$$(\omega_1)_i \begin{cases} = \omega_i & \text{if } i \notin S_1, \\ < \omega_i & \text{if } i \in S_1. \end{cases}$$

Now let $\omega_2 := (I - \theta_2 E_2 \langle M_2 \rangle^{-1} \langle A \rangle)\omega_1$. We can claim that $\omega_2 \leqslant \omega_1$ and in the components corresponding to S_2 the inequality is strict. Indeed,

$$0 \leqslant (I - \theta_2 E_2 \langle M_2 \rangle^{-1} \langle A \rangle)\omega_1 = (I - \theta_2 E_2 \langle M_2 \rangle^{-1} \langle A \rangle)(\omega_1 - \omega + \omega) \leqslant (I - \theta_2 E_2 \langle M_2 \rangle^{-1} \langle A \rangle)\omega.$$

Now, observe

$$(\omega_2)_i \begin{cases} = (\omega_1)_i \leqslant \omega_i & \text{if } i \notin S_2, \\ < \omega_i & \text{if } i \in S_2, \end{cases}$$

since $i \in S_2$ implies that

$$(\omega_2)_i = [(I - \theta_2 E_2 \langle M_2 \rangle^{-1} \langle A \rangle)(\omega_1 - \omega)]_i + (\omega - \theta_2 E_2 \langle M_2 \rangle^{-1} e)_i < \omega_i.$$

Similarly, one can show that for all $k \leqslant m - 1$,

$$(\omega_{k+1})_i \begin{cases} = (\omega_k)_i & \text{if } i \notin S_{k+1}, \\ < \omega_i & \text{if } i \in S_{k+1}. \end{cases}$$

Because $\bigcup_{i=1}^m S_i = S = \{1, 2, \ldots, n\}$, we conclude that $T_\theta \omega < \omega$. It follows that $\|T_\theta\|_\omega \leqslant \gamma$ with a scalar $\gamma \in (0, 1)$ and, therefore, $\rho(T_\theta) \leqslant \|T_\theta\|_\omega < \gamma$.

Moreover, we get

$$\|\varepsilon^{k+1}\|_\omega = \||\varepsilon^{k+1}|\|_\omega \leqslant \|T_\theta |\varepsilon^k|\|_\omega \leqslant \gamma |\varepsilon^k|\|_\omega = \gamma \|\varepsilon^k\|_\omega. \qquad \square$$

Lemma 4.5, together with the characterization (4.5) and Lemma 2.4, is the fundamental tool for proving the convergence of the multiplicative Schwarz method for LCP with an H_+-matrix.

Theorem 4.6. *Let θ_i be positive constants satisfying $0 < \theta_i \leqslant 1, i = 1, \ldots, m$. Then the sequence $\{x^k\}$ generated by Algorithm 1 converges to the solution of (1.1) for any initial point x^0. Furthermore, there exists a unique splitting $\langle A \rangle = B - C$ such that $T_\theta = B^{-1}C$ where T_θ is defined in Lemma 4.5, and this splitting is nonnegative.*

Proof. By Lemma 4.5, $\rho(T_\theta) \leqslant \|T_\theta\|_\omega < 1$ for any $\omega = \langle A \rangle^{-1} e$ with $e > 0$. Hence, Algorithm 1 converges for any initial vector x^0. Furthermore, by Lemma 2.4, there exists a unique splitting $\langle A \rangle = B - C$ such that $T_\theta = B^{-1}C \geqslant 0$.

In the following, we prove that the splitting is nonnegative, we first show that $B^{-1} = (I - T_\theta)\langle A \rangle^{-1}$ is nonnegative or, equivalently, that $B^{-1}z \geqslant 0$ for all $z > 0$. Let $v = \langle A \rangle^{-1} z \geqslant 0$. Then all we need is to show that $(I - T_\theta)v \geqslant 0$, or $T_\theta v \leqslant v$, which is proved in the same way as Lemma 4.5. Moreover, $B^{-1}C = T_\theta \geqslant 0$. Hence, the unique splitting $\langle A \rangle = B - C$ is weak regular.

To show that it is nonnegative, we need to show that $\overline{T}_\theta = CB^{-1} = I - \langle A \rangle B^{-1}$ is also nonnegative. To see this, note that $\overline{T}_\theta = (I - \overline{P}_m) \ldots (I - \overline{P}_1)$, where $\overline{P}_i = \theta_i \langle A \rangle E_i \langle M_i \rangle^{-1} = \theta_i \langle A \rangle R_i^T \langle A_i \rangle^{-1} R_i$. To complete the proof we show that each factor $I - \overline{P}_i$ is nonnegative. Since

$$\begin{aligned} I-\overline{P}_i^T &= I-\theta_i R_i^T\langle A_i\rangle^{-T}R_i\langle A\rangle^T \\ &= I-\theta_i E_i\langle M_i\rangle^{-T}\langle A\rangle^T \\ &= I-\theta_i E_i+\theta_i E_i(I-\langle M_i\rangle^{-T}\langle A\rangle^T) \\ &= I-\theta_i E_i+\theta_i E_i\langle M_i\rangle^{-T}|N_i|^T \\ &\geqslant 0, \end{aligned}$$

where the inequality follows from $\langle A\rangle=\langle M_i\rangle-|N_i|, i=1,\ldots,m$, are regular splittings. □

5. The effect of overlap on Algorithm 1

In this section, we study the effect of varying of overlap. For the classical multiplicative Schwarz method, an increase of the overlap is associated with fewer iterations. We show the similar property for Algorithm 1 by using certain weighted max-norms.

Let us consider two sets of subblocks (subdomains) of the matrix A, as defined by the sets (3.2), such that one has more overlap than the other, i.e., let

$$\widehat{S}_i\supseteq S_i, i=1,2,\ldots,m, \tag{5.1}$$

with $\bigcup_{i=1}^m\widehat{S}_i=\bigcup_{i=1}^m S_i=S$. Of course, each set $\widehat{S}_i$ defines an $\hat{n}_i\times n$ matrix $\widehat{R}_i$, where $\hat{n}_i$ is the cardinality of $\widehat{R}_i$, and the corresponding $n\times n$ matrix $\widehat{E}_i=\widehat{R}_i^T\widehat{R}_i$, as in (3.1). The relation (5.1) implies that

$$I\geqslant\widehat{E}_i\geqslant E_i\geqslant 0. \tag{5.2}$$

Similarly, we denote by $\widehat{A}_i$ the corresponding principal submatrix of A and $\hat{\pi}_i$ the corresponding permutation, respectively.

We choose a special case of M_i defined in (4.2) such that

$$B_{i_c}=D_{i_c}=\operatorname{diag}(A_{i_c}).$$

In a similar fashion, we can define a special case of $\widehat{M}_i$ such that

$$\widehat{B}_{i_c}=\widehat{D}_{i_c}=\operatorname{diag}(\widehat{A}_{i_c}).$$

We want to compare $\widehat{M}_i$ with M_i, although $\widehat{A}_i$ and A_i are of different sizes. Without loss of generality, we can assume that the permutations π_i and $\hat{\pi}_i$ coincide on the set S_i, and that the indices in S_i are the first n_i elements in $\widehat{S}_i$. In fact, we can assume that $\hat{\pi}_i=\pi_i$. Thus, A_i is a principal submatrix of $\widehat{A}_i$, and $\widehat{M}_i$ has the same diagonal as M_i. Since both $\widehat{A}_i$ and $\widehat{M}_i$ are H_+-matrices, it follows that

$$\langle\widehat{M}_i\rangle\leqslant\langle M_i\rangle,\quad i=1,2,\ldots,m. \tag{5.3}$$

We consider now the algebraic multiplicative Schwarz iteration matrix $\widehat{T}_\theta$ with larger overlap, i.e.,

$$\widehat{T}_\theta=\prod_{i=m}^{1}(I-\theta_i\widehat{E}_i\langle\widehat{M}_i\rangle^{-1}\langle A\rangle). \tag{5.4}$$

Using (5.2) and (5.3), the proof of the following theorem proceeds exactly as in the proof of Theorem 5.4 in [7]. For the sake of brevity, the proof is omitted.

Theorem 5.1. *Let θ_i be positive constants satisfying $0<\theta_i\leqslant 1$ and A be an H_+-matrix. Let T_θ and $\widehat{T}_\theta$ be the iteration matrices defined in Lemmas 4.5 and 5.4, respectively. Then $\rho(\widehat{T}_\theta)\leqslant\rho(T_\theta)$, and for any vector $\omega=\langle A\rangle^{-1}e>0$ with $e>0$, $\|\widehat{T}_\theta\|_\omega\leqslant\|T_\theta\|_\omega$.*

Remark 5.1. Theorem 5.1 indicate that the more overlap there is, the faster the convergence of Algorithm 1. As a special case, we have that overlap is better than no overlap. This is consistent with the analysis for linear systems [7]. In the case of no overlap, if $\theta_1 = \theta_2 = \cdots = \theta_m = 1$, then Algorithm 1 reduces to a block Gauss-Seidel method. Results similar to Theorem 5.1 are shown for (additive) multisplitting methods in [13,22].

6. Monotone convergence for an M-matrix

From Remark 4.1, we know that Algorithm 1 can be thought of as a multisplitting iteration method. In [1], Bai studied the monotone convergence property of the multisplitting iteration method for LCPs. So we prove in this section the monotone convergence property of Algorithm 1 when the coefficient matrix A is an M-matrix. We first recall the concept of super-solution [19]. The super-solution set of problem (1.1) is the set

$$W = \{y \in \mathbb{R}^n : y \geqslant \phi, Ay - F \geqslant 0\}. \tag{6.1}$$

This set is also called the feasible set of (1.1) in the LCP literature; see, e.g., [21]. It is well known that the solution of problem (1.1) is the least element of the super-solution W if A is an M-matrix, but it is not the case if A is an H-matrix [17]. In the following, the coefficient matrix A is always taken as an M-matrix.

Lemma 6.1. *Let $x^{k,i}$ be the solution of (2.7). If $z^{k,i-1} \in W$, then inequality $x^{k,i} \leqslant 0$ holds for each $i = 1, \ldots, m$.*

Proof. Since $Az^{k,i-1} - F \geqslant 0$ and $z^{k,i-1} \geqslant \phi$, it follows that $R_e^{k,i} \leqslant 0$ and $\phi^{k,i} \leqslant 0$ from (2.2), (2.5) and (2.6). This implies $0 \in \mathbb{R}^{n_i}$ is a super-solution of problem (2.7). Since for each $i = 1, \ldots, m$, A_i is also an M-matrix, it follows that $x^{k,i}$ is the least element of the super-solution of (2.7) and hence we have $x^{k,i} \leqslant 0$. $\square$

Lemma 6.2. *If $0 < \theta_i \leqslant 1$ for each $i = 1, \ldots, m$, and $z^{k,i-1} \in W$, then $z^{k,i} \in W$.*

Proof. Let R_i and E_i be defined by (2.2) and (3.1), respectively. By (2.6) and (2.7), we have $x^{k,i} \geqslant R_i(\phi - z^{k,i-1})$. It then follows that

$$\begin{aligned} z^{k,i} &= z^{k,i-1} + \theta_i R_i^T x^{k,i} \\ &\geqslant z^{k,i-1} + \theta_i R_i^T R_i(\phi - z^{k,i-1}) \\ &= z^{k,i-1} + \theta_i E_i(\phi - z^{k,i-1}) \\ &= \phi + (I - \theta_i E_i)(z^{k,i-1} - \phi) \\ &\geqslant \phi, \end{aligned}$$

where the last inequality follows from $0 \leqslant E_i \leqslant I$ and $0 < \theta_i \leqslant 1$. Since the equalities $\pi^T\pi = \pi\pi^T = I$ hold for any permutation matrix π. We get

$$\begin{aligned} Az^{k,i} - F &= A(z^{k,i-1} + \theta_i R_i^T x^{k,i}) - F \\ &= Az^{k,i-1} - F + \theta_i A \pi_i^T \begin{bmatrix} I_i \\ 0 \end{bmatrix} x^{k,i} \\ &= Az^{k,i-1} - F + \theta_i \pi_i^T A_{\pi_i} \begin{bmatrix} I_i \\ 0 \end{bmatrix} x^{k,i} \\ &= Az^{k,i-1} - F + \theta_i \pi_i^T \begin{bmatrix} A_i \\ H_i \end{bmatrix} x^{k,i}, \end{aligned}$$

where the last equality follows from (2.4). For each $i = 1, \ldots, m$, A_{π_i} is an M-matrix, which implies $H_i \leqslant 0$. It follows from Lemma 6.1 that

$$
\begin{aligned}
Az^{k,i} - F &\geqslant Az^{k,i-1} - F + \theta_i \pi_i^T \begin{bmatrix} A_i \\ 0 \end{bmatrix} x^{k,i} \\
&= Az^{k,i-1} - F + \theta_i \pi_i^T \begin{bmatrix} I_i \\ 0 \end{bmatrix} A_i x^{k,i} \\
&= Az^{k,i-1} - F + \theta_i R_i^T A_i x^{k,i} \\
&\geqslant Az^{k,i-1} - F + \theta_i R_i^T R_e^{k,i} \\
&= Az^{k,i-1} - F + \theta_i R_i^T R_i (F - Az^{k,i-1}) \\
&= Az^{k,i-1} - F + \theta_i E_i (F - Az^{k,i-1}) \\
&= (I - \theta_i E_i)(Az^{k,i-1} - F) \\
&\geqslant 0,
\end{aligned}
$$

where the second inequality follows from (2.7), and the third inequality follows from the condition $z^{k,i-1} \in W$ and $I - \theta_i E_i \geqslant 0$. Inequality $z^{k,i} \geqslant \phi$ together with inequality $Az^{k,i} - F \geqslant 0$ implies $z^{k,i} \in W$. $\square$

The following theorem shows monotone convergence of Algorithm 1 when the coefficient matrix A is an M-matrix.

Theorem 6.3. *Let θ_i be positive constants satisfying $0 < \theta_i \leqslant 1, i = 1, \ldots, m$. If $x^0 \in W$, then the sequence $\{x^k\}$ generated by Algorithm 1 converges to the solution $\bar{x}$ of (1.1). Moreover, we have for any $k \geqslant 0$*

$$x^k \in W \quad \text{and} \quad \bar{x} \leqslant x^{k+1} \leqslant x^k. \tag{6.2}$$

Proof. Since $x^0 \in W$, it follows that $x^k \in W$ holds from the deduce of Algorithm 1 and Lemma 6.2. Moreover, it follows from (2.8), (2.9) and Lemma 6.1 that $x^{k+1} \leqslant x^k$ for all $k \geqslant 0$. In particular, $\{x^k\}$ converges to $\bar{x}$ from Theorem 4.6. Clearly, we have $\bar{x} \leqslant x^k$ for all $k \geqslant 0$. $\square$

7. Numerical results

In this section, we give some numerical experiments to investigate the behavior of the algorithm presented in this paper. The program is coded in Visual C++ 6.0 and run on a personal computer with 1.8GHz CPU and 244MB memory. We consider the linear complementarity problem with the system matrix and given vectors:

$$
A = \begin{pmatrix}
S & -I & 0 & 0 & \cdots & 0 & 0 \\
-I & S & -I & 0 & \cdots & 0 & 0 \\
0 & -I & S & -I & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & & \ddots & \ddots & S & -I \\
0 & 0 & \cdots & \cdots & 0 & -I & S
\end{pmatrix} \in \mathbb{R}^{n \times n}, \quad
F = \begin{pmatrix} -1 \\ 1 \\ -1 \\ \vdots \\ (-1)^{n-1} \\ (-1)^n \end{pmatrix} \in \mathbb{R}^n
$$

and $\phi = 0$, respectively, where $S = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{\bar{n} \times \bar{n}}$, I is the identity matrix, and $\bar{n}^2 = n$, see [5,6]. It is known that A is an H_+-matrix and, therefore, the above LCP has a unique solution.

We have conducted the following experiments: (a) comparing Algorithm 1 with project successive overrelaxation (PSOR) algorithm [16]; (b) testing Algorithm 1 with different overlapping sizes.

Table 1
Effect of relaxation parameter ω in PSOR algorithm.

ω	Iter	CPU	ω	Iter	CPU
1.0	1013	28.593	1.1	848	23.953
1.2	708	20	1.4	477	13.5
1.6	291	8.296	1.8	128	3.75
1.84	91	2.688	1.85	84	2.515
1.86	87	2.593	1.9	99	2.89

Table 2
Comparison of iteration numbers.

n	PSOR	$O\left(\frac{1}{10}\right)$	$O\left(\frac{1}{2}\right)$
100	52	7	4
400	57	10	4
900	72	12	5
1600	84	14	6
2500	147	16	9
3600	223	21	11
4900	309	22	15

First, we use PSOR algorithm to solve the LCP with $n = 1600$. In this test, we mainly consider iteration numbers (denoted by Iter) and execution times (denoted by CPU). Numerical results are listed in Table 1. From Table 1, we may see that the optimal factor is $\omega = 1.85$.

In testing Algorithm 1 with different overlapping sizes, we partition $\mathbb{R}^n$ into two equal parts with the overlapping size $O\left(\frac{1}{10}\right)$ and $O\left(\frac{1}{2}\right)$, respectively. And the corresponding subproblems are solved by PSOR algorithm with the same relaxation parameter $\omega = 1.85$. We choose initial vector $x^0 = (5, 5, \dots, 5)^T$ and the weights $\theta_1 = \theta_2 = 1$. The tolerance in the subproblems of the solution algorithms is chosen to be equal to 10^{-6} in the $\|\cdot\|_2$-norm, and in the outer iterative processes is chosen to be equal to 10^{-6} in the $\|\cdot\|_2$-norm.

Table 2 gives the iteration history for the above iterative methods and Table 3 reports the execution times. From the tables, we can see that

(1) Algorithm 1 is better than PSOR algorithm in iteration numbers. With the growth in the dimension n of the LCP, the iterative number of Algorithm 1 changes slightly while PSOR algorithm increases quickly.
(2) the convergence of Algorithm 1 is faster when the overlapping is larger.

Table 3
Comparison of execution times (CPU seconds).

n	PSOR	$O\left(\frac{1}{10}\right)$	$O\left(\frac{1}{2}\right)$
100	0.015	0.015	0.015
400	0.109	0.406	0.296
900	0.656	2.281	1.531
1600	2.515	7.812	5.453
2500	10.171	21.75	18.656
3600	31.328	81.125	379.125
4900	103.203	1788.77	2393.36

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